Computing Teichmüller Space Coordinates

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Thank for the invitation.
The work is collaborated with Shing-Tung Yau, Feng Luo, Miao Jin and many other mathematicians, computer scientists and doctors.
Isothermal Coordinates Relation between conformal structure and Riemannian metric

**Isothermal Coordinates**

A surface $\Sigma$ with a Riemannian metric $g$, a local coordinate system $(u,v)$ is an isothermal coordinate system, if

$$g = e^{2\lambda(u,v)}(du^2 + dv^2).$$
Suppose $\tilde{g} = e^{2\lambda} g$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$K = -\Delta_g \lambda = -\frac{1}{e^{2\lambda}} \Delta \lambda,$$

where

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$$
Conformal Metric Deformation

**Definition**

Suppose $\Sigma$ is a surface with a Riemannian metric,

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}g$ is also a Riemannian metric on $\Sigma$ and called a **conformal metric**. $\lambda$ is called the conformal factor.

$$g \rightarrow e^{2\lambda}g$$

Conformal metric deformation.

Angles are invariant measured by conformal metrics.
Yamabi Equation

Suppose $\bar{g} = e^{2\lambda} g$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2\lambda} (-\Delta_g \lambda + K),$$

godesic curvature on the boundary

$$\bar{k}_g = e^{-\lambda} (-\partial_n \lambda + k_g).$$
Uniformization

**Theorem (Poincaré Uniformization Theorem)**

Let $(\Sigma, g)$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{g} = e^{2\lambda} g$ conformal to $g$ which has constant Gauss curvature.
Key Idea

\[ K = -\Delta_g \lambda, \]

Roughly speaking,

\[ \frac{dK}{dt} = \Delta_g \frac{d\lambda}{dt} \]

Let \( \frac{d\lambda}{dt} = -K \),

\[ \frac{dK}{dt} = \Delta_g K \]

Heat equation!
Definition (Hamilton’s Surface Ricci Flow)

A closed surface with a Riemannian metric $g$, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = -Kg_{ij}.$$ 

If the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant everywhere.
Ricci Flow

**Theorem (Hamilton 1982)**

For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) everywhere.

**Theorem (Bennett Chow)**

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) everywhere.
Summary

Surface Ricci Flow

- Conformal metric deformation

\[ g \rightarrow e^{2u} g \]

- Curvature Change - heat diffusion

\[ \frac{dK}{dt} = \Delta_g K \]

- Ricci flow

\[ \frac{du}{dt} = \bar{K} - K. \]
Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
  - Isometric gluing of triangles in $\mathbb{E}^2$.
  - Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$. 

![Image of triangular mesh models]
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![Image of generic surface models](image-url)
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Discrete Generalization

Concepts

1. Discrete Riemannian Metric
2. Discrete Curvature
3. Discrete Conformal Metric Deformation
Discrete Metrics

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, \( I : E = \{ \text{all edges} \} \to \mathbb{R}^+ \), satisfies triangular inequality.

A mesh has infinite metrics.
Discrete Curvature

Definition (Discrete Curvature)

Discrete curvature: $K : V = \{\text{vertices}\} \to \mathbb{R}^1$.

$$K(v) = 2\pi - \sum_i \alpha_i, \; v \notin \partial M; \quad K(v) = \pi - \sum_i \alpha_i, \; v \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi \chi(M).$$
Discrete Metrics Determines the Curvatures

\[ \cos l_i = \frac{\cos \theta_i + \cos \theta_j \cos \theta_k}{\sin \theta_j \sin \theta_k} \]  \hspace{1cm} (1)

\[ \cosh l_i = \frac{\cosh \theta_i + \cosh \theta_j \cosh \theta_k}{\sinh \theta_j \sinh \theta_k} \]  \hspace{1cm} (2)

\[ 1 = \frac{\cos \theta_i + \cos \theta_j \cos \theta_k}{\sin \theta_j \sin \theta_k} \]  \hspace{1cm} (3)
Discrete Conformal Factor for Yamabe Flow

\[
\begin{align*}
\frac{y_k}{2} &= e^{u_i} l_k \ e^{u_j} \\
\sinh \frac{y_k}{2} &= e^{u_i} \sinh \frac{l_k}{2} \ e^{u_j} \\
\sin \frac{y_k}{2} &= e^{u_i} \sin \frac{l_k}{2} \ e^{u_j}
\end{align*}
\]

Properties: \( \frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i} \) and \( dK = \Delta dU \).
Unified Framework of Discrete Curvature Flow

**Analogy**

- **Curvature flow**
  \[
  \frac{du}{dt} = \bar{K} - K,
  \]

- **Energy**
  \[
  E(u) = \int \sum_i (\bar{K}_i - K_i) du_i,
  \]

- **Hessian of** $E$ **denoted as** $\Delta$,
  \[
  dK = \Delta du.
  \]
Conformal Mapping preserves angles.

Riemann Mapping
Conformal Mapping

Definition (Conformal Mapping)
Suppose \((S_1, g_1)\) and \((S_2, g_2)\) are two metric surfaces, \(\phi : S_1 \to S_2\) is conformal, if on \(S_1\)
\[
g_1 = e^{2\lambda} \phi^* g_2,
\]
where \(\phi^* g_2\) is the pull-back metric induced by \(\phi\).

Definition (Conformal Equivalence)
Suppose two surfaces \(S_1, S_2\) with marked homotopy group generators, \(\{a_i, b_i\}\) and \(\{\alpha_i, \beta_i\}\). If there exists a conformal map \(\phi : S_1 \to S_2\), such that
\[
\phi_*[a_i] = [\alpha_i], \phi_*[b_i] = [\beta_i],
\]
then we say two marked surfaces are conformal equivalent.
Definition (Teichmüller Space)

Fix the topology of a marked surface $S$, all conformal equivalence classes sharing the same topology of $S$, form a manifold, which is called the Teichmüller space of $S$. Denoted as $T_S$. 
Each point represents a class of surfaces.
A path represents a deformation process from one shape to the other.
The Riemannian metric of Teichmüller space is well defined.
Conformal Module

Topological Quadrilateral - 1D
Conformal Module

Topological Annulus - 1D
Conformal Module

Multiply Connected Domains - Möbius Transformation - 3D
Koebe
Conformal Module

Multiply Connected Domains - (3n-3) D
Conformal Module

Multiply Connected Domains
Conformal Module

Multiply Connected Domains

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Teichmüller Space
Conformal Module

Multiply Connected Domains
Conformal Module

Genus One Surfaces - 2D
Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.
Hyperbolic Uniformization Metric

Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.
Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.
Embedding in the upper half plane hyperbolic space model. Different period embedded in the hyperbolic space. The boundaries are mapped to hyperbolic lines.
The Teichmüller coordinates (conformal module) are given by the hyperbolic lengths of \((\gamma_i, \gamma_j, \gamma_k)\). \(\tau_i\) is the shortest path connecting \(\gamma_j, \gamma_k\), which is a geodesic orthogonal to \(\gamma_j, \gamma_k\).
Conformal Module

Topological Pants - 3D

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Teichmüller Space
Shape Space

Teichmüller Space
Each homotopy class has a unique closed geodesic on a surface with negative curvature metric.
Canonical Homotopy Class Representative

Geodesic

- The rigid motion of $\mathbb{H}^2$ is a Möbius transformation
  \[
  z \rightarrow e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}
  \]
- The homotopy class is isomorphic to Deck transformation group
- Deck transformations are rigid motions under hyperbolic metric
- Each homotopy class corresponds to a Möbius transformation
- The unique closed geodesic is the axis of the corresponding Möbius transformation.
Conformal Module

Topological Pants Decomposition - $2g - 2$ pairs of Pants
Conformal Module

Topological Pants Decomposition - $2g - 2$ pairs of Pants

Denh twist.
Conformal Module

Topological Genus Two Surface - $6g - 6D$
Conformal Module

Topological Genus Two Surface
Conformal Module
Teichmüller Coordinates
Teichmüller Coordinates
Teichmüller Coordinates
Teichmüller Coordinates
Conclusion

- Euclidean and Hyperbolic Yamabe curvature flow
- Conformal Module for multiply connected domains
- Teichmüller coordinates for high genus surfaces
For more information, please email to gu@cs.sunysb.edu.

Thank you!